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The scenario that the Universe contracts towards a big crunch and then undergoes a transition to expanding Universe in envisaged in the quantum string cosmology approach. The Wheeler-De Witt equation is solved exactly for an exponential dilaton potential. S-duality invariant cosmological effective action, for type IIB theory, is considered to derive classical solutions and solve WDW equations.
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It is natural to expect that cosmology must be ultimately founded on quantum gravity. Since string theory/M-theory synthesise quantum mechanics and general theory of relativity; therefore, the evolution of the Universe in early epochs and fundamental issues concerned with initial singularity may be resolved in the frame work of string theory. In recent years, there has been considerable amount of interest in cosmology from the string theory point of view [1,2]. The pre-big bang (PBB) scenario [3], which has drawn a lot of attention, proposes an alternative mechanism for inflation when contrasted with the original paradigm of inflation [4,5] and promises a possible formulation of nonsingular cosmology. One of the postulates of PBB cosmology is that the Universe, in the remote past, $t \to -\infty$, is described by weak coupling, low curvature and cold state and it undergoes an accelerated expansion by the kinetic energy term of the dilaton while proceeding towards the singularity lying in its future. Subsequently, there is a transition from the accelerating to the FRW like branch in the $t > 0$ region. However, one encounters no-go theorems for the branch change while dealing with the tree level effective action [6,7]. There are several other attempts to understand diverse aspects of cosmology in the frame work of string theory [8–11].

Recently, Khoury et al. [12] have put forward a proposal where the Universe contracts towards a big crunch and then makes a transition to an expanding big bang Universe. The scenario envisaged in [12] holds the promise to explore new class of cosmological models. Furthermore, it has been pointed out that this idea leads to interesting connection with recently proposed ekpyrotic model [13] which has generated considerable activities [14]. The essential ingredients in ref. [12] is to consider an effective action with graviton and a massless scalar field, dilaton, describing the evolution of the Universe. This model incorporates some of the ideas of PBB proposal in that the evolution of the Universe began in the far past. However; it also differs from the scenario of the PBB that the Universe followed the accelerated expanding branch for $t < 0$ and then it exits to the expanding, decelerating branch for which the singularity lies in its past.

The purpose of this note is to present an investigation of the mechanism for the transition from big crunch to big bang from the quantum mechanical perspective. We derive the Wheeler-De Witt equation for the case at hand and impose appropriate boundary conditions which describes the emergence of the FRW type Universe starting from initial state which corresponds to big crunch classically. To be specific, we adopt an exponential potential and choose a suitable metric to facilitate solution for the case under considerations. We note that *raison de etre* for the exponential potentials have been argued by Moore, Peradze and Salina [15] from M-theoritic analysis. Furthermore, exponential potentials arising from M-theory cosmology might explain quintessence as expounded in [16].

The $D = 4$ tree level string effective action is

$$S = \int d^4x\sqrt{-g}\left(\mathcal{R}_g - \frac{1}{2}(\partial\tilde{\phi})^2 - V(\tilde{\phi})\right) \tag{1}$$

Here $g_{\mu\nu}$ is the Einstein frame metric, g is its determinant and $\mathcal{R}_g$ is the Ricci scalar derived from $g_{\mu\nu}$. Note that $\tilde{\phi}$ is the dilaton and $V(\tilde{\phi})$ is the dilaton potential term. The metric and dilaton are taken to be time dependent in the cosmological scenario and we shall consider isotropic, homogeneous and spatially flat FRW metric. In ref [12], the following form of metric was adopted: $ds^2 = a(\tau)^2(-N(\tau)^2 d\tau^2 + \sum_{i=1}^{3} dx_i^2)$, τ is conformal time. We shall choose a different form of metric [17]

$$ds^2 = -\frac{N(t)^2}{a(t)^2} + a(t)^2\delta_{ij}dx^i dx^j \tag{2}$$

Here $N(t)$ is the lapse function. It is easy to show that $\sqrt{-g}\mathcal{R}_g = -6N(t)^{-1}a(t)^2\dot{a(t)}^2$ + total derivative term. It is very useful for our purpose [17] to rescale $a(t)$ and $\tilde{\phi}$ to ϕ to bring the action to more convenient form

$$L = N(t)^{-1}\left(-\frac{1}{2}a^2\dot{a}^2 + \frac{1}{2}a^4\dot{\phi}^2\right) - Na^2V(\phi) \tag{3}$$

where the dot denotes time derivative. The equation of motion for N leads to the Hamiltonian constraint and the equation of motion for a and ϕ can be derived easily. We choose an exponential potential i.e. $V(\phi) = V_0 e^{\alpha\phi}$, V_0 being a constant. The motivation for choosing the exponential dilaton potential has been alluded to earlier. We look for solutions where scale factor, a, has a power law growth in time: $a(t) = |t|^p, \quad p > 0$. The Hubble parameter, H, its time derivative, the dilaton, ϕ and its time derivative have the following form as can be inferred from the equations of motion.

$$H = \frac{\dot{a}}{a} = \frac{p}{t}, \quad \dot{H} = -\frac{p}{t^2} \tag{4}$$

$$\phi = \phi_0 + \sqrt{\frac{(p(1-p)}{3}} ln|t|, \quad \dot{\phi} = \sqrt{\frac{p(1-p)}{3}} \frac{1}{t} \tag{5}$$

Here a_0 and ϕ_0 are arbitrary constants. The parameter α appearing in the potential gets fixed in terms of exponent p and so is the ratio of V_0 and a_0^2,

$$\alpha = -2\sqrt{\frac{3(1-p)}{p}}, \quad \text{and} \quad \frac{V_0}{a_0^2} = \frac{4p^2 - p}{6} \tag{6}$$

Cosmological solutions with exponential potentials, for different form of metric, has been considered in the past [18] and the have been topics of discussion more recently in the context of ekpyrotic model [19]. Note that, G_N or the Planck mass, m_P does not appear in our action and therefore, these constants are also absent in our solutions of equations of motion. From now on we *choose* $\alpha = -2$ so that

$$V = V_0 e^{-2\phi}, \quad \text{correspondingly} \quad p = \frac{3}{4} \tag{7}$$

Note that we have also absorbed a factor of $e^{-2\phi_0}$ in the definition of V_0.
Let us define, new set of variables: $u = \frac{a^2}{2}\cosh 2\phi$ and $v = \frac{a^2}{2}\sinh 2\phi$; then the Lagrangian is

$$L = \frac{1}{2N}\left(\dot{v}^2 - \dot{u}^2\right) - \frac{N}{2} V_0 (u - v) \tag{8}$$

for our choice of specific exponential potential. The corresponding Hamiltonian is

$$\mathcal{H} = N(\frac{1}{2}p_v^2 - \frac{1}{2}p_u^2 + \frac{V_0}{2}[u - v]) \tag{9}$$

varying N yields the constraint $H = 0$ in $N = 1$ gauge. The momenta p_v and p_u are conjugate to v and u respectively. Furthermore, $[p_u + p_v, \mathcal{H}]_{PB} = 0$, implying existence of a conserved charge,

$$Q = \frac{e^{-2\phi}}{a(t)^2}[-H + \dot{\phi}^2] \tag{10}$$

H is the Hubble parameter. Note that $\phi = \phi_0 + \frac{1}{4} ln|t|$, and therefore, Q contains a factor $e^{-2\phi_0}$ in its definition, revealing the coupling constant dependence.
Now we obtain the Wheeler-De Witt equation and impose appropriate boundary conditions on the wave function. It has been advocated that quantum string cosmology might be useful to address the issue of graceful exit [20–22]and to study evolution of the early Universe [23].

The Wheeler-De Witt equation takes the following form for the Hamiltonian (set $N = 1$) (9),

$$[-\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial u^2} + \frac{V_0}{2}(u - v)]\psi(v, u) = 0 \tag{11}$$

We may solve (11) by separation of v and u and $\psi(u, v)$ is product of two Airy functions. However, it is useful to reexpress the WDW equation in terms of another set of variables and the wave function thus derived has a more direct intepretation in terms of a scenario that the Universe evolved towards (classical path of) big crunch in negative t region and then undergoes a transition to the expanding phase for $t > 0$. Define,

$$\xi = \frac{1}{6} ln 4XY, \quad \zeta = \frac{1}{6} ln\frac{X}{Y} \tag{12}$$

with $X = \frac{1}{4}(u + v)$ and $Y = \frac{1}{8}(u - v)^2$. The Wheeler-De Witt equation is expressed as (in the gauge $N^{-1} = \frac{1}{2}[u - v]$)

$$\mathcal{H}\psi(\xi, \zeta) = \left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \zeta^2} + 9V_0 e^{6\xi}\right)\psi(\xi, \zeta) = 0 \tag{13}$$

The wave function is a product of a plane wave in ζ variable of the form $e^{\pm ik\zeta}$, where k is the separation constant. Note that k is also identified with eigenvalue of the momentum operator $i\frac{\partial}{\partial \zeta}$ acting on the plane wave solution (see the arguments in [20] for the choice of the sign in defining operator p_ζ) and this is a constant of motion since $[p_\zeta, H] = 0$. Indeed, the conserved momentum is related to the charge Q, defined in (10). The solution to the equation of ξ variable is Bessel function; therefore, ψ is given by

$$\psi_k(\xi, \zeta) = e^{\pm ik\zeta}\mathcal{F}_{\pm\frac{ik}{3}}(z), \quad z = \sqrt{V_0}e^{3\xi} \tag{14}$$

where $\mathcal{F}_\nu(z)$ is one of the Bessel functions, $J_\nu(z)$, $Y_\nu(z)$ or Hankel functions $H_\nu^{(1,2)}(z)$. The relevant Bessel function is chosen in order to fulfill desired boundary conditions. We demand that there is only right moving wave in the negative t region, with positive eigenvalue of p_ξ, and are led to choose $J_{-i\frac{k}{3}}(z)$. In this domain, for small geometries, as $a \to 0$, $z \to 0$ and

$$\lim_{z\to 0} \quad J_{-i\frac{k}{3}}(z) \sim \frac{1}{2}(z)^{-i\frac{k}{3}} \sim e^{-ik\xi} \tag{15}$$

Thus, in this limit, the wave function behaves as

$$\psi_k(\xi, \zeta) \sim e^{-ik(\xi+\zeta)} \tag{16}$$

The Universe expands in the $t > 0$ region. The scale factor grows with time and we are led to consider behavior of ψ when $\xi \to \infty$. The asymptotic form of ψ is expressed as sum of two components:

$$\psi_k(\xi, \zeta) = e^{-ik\zeta} J_\nu(z) =_{\xi\to\infty} \psi_{k(+)} + \psi_{k(-)} \tag{17}$$

where $\nu = -i\frac{k}{3}$ and the two components are given by

$$\psi_{k(\pm)} = \sqrt{\frac{2}{\pi z}} e^{-ik\zeta} e^{\mp i(z - \frac{\pi}{2} - \frac{1}{2}\nu\pi - \frac{\pi}{4})} \tag{18}$$

Moreover, $\psi_{k(\pm)}$ satisfy following relations when operated up on by the momentum operator, $p_\xi = -i\frac{\partial}{\partial\xi}$

$$p_\xi \psi_{k(\pm)} = \mp z \psi_{k(\pm)} \tag{19}$$

Thus, the choice of the wave function describes the evolution of the Universe as follows: the right moving wave propagates from the -ve t region. The wave function in the positive time domain is superposition of the left and right moving components. The boundary conditions adopted to choose the vave function is in the spirit of Vilenkin's proposal [24]. The probability for the transition from the branch with wave function of the form $e^{-ik\zeta - ikz}$ to the branch which has wave function $e^{-ik\zeta + ikz}$, in the asymptotic limit, is given by

$$P_k = \frac{|\psi_{k(-)}|^2}{|\psi_{k(+)}|^2} = e^{-\frac{2}{3}\pi k} \tag{20}$$

Recall that k is related to the conserved charge (10) which contains a factor of $e^{-2\phi_0}$ in its definition. Thus the probability gets suppressed for in the weak coupling phase.

Next, we proceed to discuss cosmological solutions, in the present context, for type IIB effective action. The dilaton, ϕ and RR scalar, χ, called axion belong to $SL(2,R)$ S-duality group and they parametrize the coset $\frac{SL(2,R)}{SO(2)}$. The 10-dimensional effective action can expressed in manifestly S-duality invariant form [25] in the Einstein frame. The toroidal compactification of that action to lower dimensions, preserving S-duality invariance, was presented by the author and Roy [26,27] which has been useful to obtain classical solutions of IIB theory. We consider a simple version of the 4-dimensional type IIB effective action: the compactification radii of the tori are taken to be constant (set to one), only graviton, dilaton and axion are retained in the reduced action and furthermore, rest of the scalar, vector and tensor fields are set to zero. We refer the interested reader to [26] where the full reduced action is derived. Our starting point is

$$S_4 = \int d^4x \sqrt{-g} \left(\mathcal{R}_g + \frac{1}{4} \mathrm{Tr}[\partial_\mu \mathcal{M} \Sigma \partial^\mu \mathcal{M} \Sigma] \right) \tag{21}$$

This is the action in the Einstein frame with,

$$\mathcal{M} = \begin{pmatrix} \chi^2 e^\phi + e^{-\phi} & \chi e^\phi \\ \chi e^\phi & e^\phi \end{pmatrix}, \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{22}$$

Here Σ is the SL(2,R) metric in our conventions [26]. The above action is manifestly invariant under S-duality transformations

$$\mathcal{M} \to \Lambda \mathcal{M} \Lambda^T, \quad g_{\mu\nu} \to g_{\mu\nu}, \quad \Lambda \Sigma \Lambda^T = \Sigma \tag{23}$$

where $\Lambda \in SL(2,R)$ with unit determinant. Some of the useful relations these matrices satisfy are

$$\Sigma \Lambda \Sigma = \Lambda^{-1}, \mathcal{M} \Sigma \mathcal{M} = \Sigma, \quad and \quad \Sigma \mathcal{M} \Sigma = \mathcal{M}^{-1} \tag{24}$$

Note that $\mathcal{M} \in SL(2,R)$ and is symmetric. For the cosmological case, with our form of FRW metric (after some rescaling) the Langrangian is

$$L = -\frac{1}{2N} a^2 \dot{a}^2 - \frac{a^4}{4N} \mathrm{Tr}[\dot{\mathcal{M}} \Sigma \dot{\mathcal{M}} \Sigma] \tag{25}$$

$\mathcal{M}$-equation of motion deserves some care, since $\mathcal{M} \in SL(2,R)$ and is a conservation law as expected [26,28].

$$\partial_0 (\sqrt{-g} g^{00} \mathcal{M} \Sigma \partial_0 \mathcal{M} \Sigma) = 0 \tag{26}$$

and thus we conclude $\sqrt{-g} g^{00} \mathcal{M} \Sigma \partial_0 \mathcal{M} = A$, where A is a constant 2×2 matrix; is symmetric and satisfies $A \Sigma \mathcal{M} = -\mathcal{M} \Sigma A$ which follow by using the relations between $\mathcal{M}$ and Σ mentioned above. Another important relation is (using $g^{00}\sqrt{-g} = a^4$)

$$\mathrm{Tr}(\dot{\mathcal{M}} \Sigma \dot{\mathcal{M}} \Sigma) = -\frac{1}{a^8} \mathrm{Tr}(A \Sigma A \Sigma) \tag{27}$$

The time dependence in the *r.h.s.* is buried in the presence of a. The Hamiltonian constraint relates the Hubble parameter and scale factor appearing through relation (27)

$$H^2 = \frac{1}{2a^8} \mathrm{Tr}(A \Sigma A \Sigma) \tag{28}$$

resulting in the time dependence: $a = a_0 t^{\frac{1}{4}}$, a_0 being a constant including the factor coming from $\mathrm{Tr} A \Sigma A \Sigma$. Notice that the Einstein-Friedman equation derived from (25) is also satisfied when (28) solved. We can solve (26) once $a(t)$ is determined

$$\mathcal{M}(t) = e^{A \Sigma ln(t - t_0)} \mathcal{M}(t_0) \tag{29}$$

where t_0 is an arbitrary constant and $\mathcal{M}(t_0)$ is value of the matrix at t_0. We may argue that graviton-axion-dilaton system effectively introduces a dilaton potential. The kinetic energy term of axion-dilaton part is: $\frac{1}{2}(a^4 \dot{\phi}^2 + a^4 e^{2\phi} \dot{\chi}^2)$ The axion equation of motion is a conservation law with a conserved charge, $Q_a = a^4 e^{2\phi} \dot{\chi}$. Therefore, while solving coupled equations of motion, we may eliminate $\dot{\chi}$ and a term resembling exponential dilaton potential appears.

Let us derive the WDW equation for the $SL(2,R)$ invariant system. First of all define $a(t) = e^{\alpha(t)}$ and then a new time variable through the relation $d\tau = e^{-4\alpha} dt$ and the derivatives with respect to τ are denoted by prime.

$$L = \int d\tau [-\frac{1}{2}\alpha'^2 - \frac{1}{4}\mathrm{Tr}(\mathcal{M}'\Sigma\mathcal{M}'\Sigma)] \tag{30}$$

α' appearing here not to be confused with inverse string tension. The canonical momenta are

$$p_\alpha = -\alpha', \quad \Pi_{\mathcal{M}} = -\frac{1}{2}\Sigma\mathcal{M}'\Sigma \tag{31}$$

The canonical Hamiltonian is

$$\mathcal{H} == -\frac{1}{2}p_\alpha^2 - \mathrm{Tr}(\Sigma\Pi_{\mathcal{M}}\Sigma\Pi_{\mathcal{M}}) \tag{32}$$

The WDW equation assumes the following form

$$\left(\frac{\delta^2}{\delta\alpha^2} + \mathrm{Tr}(\Sigma\frac{\delta}{\delta\mathcal{M}}\Sigma\frac{\delta}{\delta\mathcal{M}})\right)\Psi(\alpha,\mathcal{M}) = 0 \tag{33}$$

We may factorize $\Psi(\alpha,\mathcal{M}) = \mathcal{F}(\alpha)\mathcal{G}(\mathcal{M})$. From the conservation law of the $\mathcal{M}$-matrix evolution equation, the quantum mechanical relation

$$\mathcal{M}\Pi_{\mathcal{M}}\mathcal{G} = i\mathcal{M}\frac{\delta}{\delta\mathcal{M}}\mathcal{G} = (2A\Sigma)\mathcal{G} \tag{34}$$

follows immediately. The wave equation satisfied by $\mathcal{F}$ is

$$\left(\frac{\delta^2}{\delta\alpha^2} + \frac{1}{4}\mathrm{Tr}(A\Sigma)^2\right)\mathcal{F}(\alpha) = 0 \tag{35}$$

which leads to a 'plane wave' solution

$$\mathcal{F}(\alpha) = e^{\pm i\frac{\alpha}{2}[\mathrm{Tr}(A\Sigma)^2]^{\frac{1}{2}}} \tag{36}$$

To solve for $\mathcal{G}$ with constraint (35), one needs to specify the matrix A. For example, when $\chi = 0$, $\mathcal{M}$ is diagonal and $\mathcal{G}$ is pure dilatonic plane wave. More general form of A will lead to interesting class of solutions respecting S-duality. An important point is that an arbitrary potential $V(\phi)$ added to Langrangian (25) breaks S-duality symmetry; moreover, the choice of S-duality invariant potentials are severely restricted [26]. Therefore, such symmetry considerations might play important roles in the study of cosmological solutions [29].

In summary, we have presented a quantum string cosmological investigation of the scenario that the intial state of the Universe is the one which evolves towards big crunch in negative regime and subsequently, the wave function gets reflected. The probability is exponetially suppressed in the weak coupling regime. Furthermore, we considered an S-duality invariant action and solved the classical equations in a general setting and discussed the structures of the wave functions of the corresponding WDW equations.

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